

## WEAK CHEBYSHEV SUBSPACES AND $A$ -SUBSPACES OF $C[a, b]$

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**ABSTRACT.** In this paper we show some very interesting properties of weak Chebyshev subspaces and use them to simplify Pinkus's characterization of  $A$ -subspaces of  $C[a, b]$ . As a consequence we obtain that if the metric projection  $P_G$  from  $C[a, b]$  onto a finite-dimensional subspace  $G$  has a continuous selection and elements of  $G$  have no common zeros on  $(a, b)$ , then  $G$  is an  $A$ -subspace.

### 1. INTRODUCTION

Let  $K$  be a compact subset of  $R^n$  ( $n \geq 1$ ) satisfying  $K = \overline{\text{int } K}$  (the closure of  $\text{int } K$ ) and denote by  $C_\omega(K)$  the space of all real-valued continuous functions on  $K$  endowed with the  $\omega$ -weighted  $L_1$ -norm:

$$\|f\|_\omega := \int_K \omega |f| d\mu,$$

where  $\mu$  denotes the Lebesgue measure. Let

$$W_\infty := \{\omega \in L_\infty(K, \mu) : \omega > 0 \text{ on } K\}.$$

For a subspace  $G$  of  $C(K)$ , the space of all real-valued continuous functions on  $K$  endowed with the supremum norm, let

$$G^* := \{g^* \in C(K) : |g^*| \equiv |g| \text{ for some } g \in G\}.$$

We say that a finite-dimensional subspace  $G$  of  $C(K)$  satisfies the  $A$ -property (or is an  $A$ -subspace), if for every  $g^* \in G^* \setminus \{0\}$  there exists  $g \in G \setminus \{0\}$  such that  $g = 0$  a.e. on  $Z(g^*)$  (the set of all zeros of  $g^*$ ) and  $g \cdot g^* \geq 0$  on  $K$ . Let  $W$  be a convex cone in  $W_\infty$  satisfying the following condition:

if  $p$  is a bounded, measurable function and  $\int_K \omega \cdot p d\mu \geq 0$  for all  $\omega \in W$ , then  $p \geq 0$  on  $K$ .

Then we have the following characterization of Chebyshev subspaces of  $C_\omega(K)$  with respect to varying weights.

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**Theorem 1.1.** *Suppose that  $G$  is a finite-dimensional subspace of  $C(K)$ . Then  $G$  is a Chebyshev subspace of  $C_\omega(K)$  for every  $\omega \in W$  if and only if  $G$  is an  $A$ -subspace.*

This theorem has a long history. Jackson [5], Krein [7], Galkin [3], Carroll, Braess [1], and Sommer [16] showed that the spaces of algebraic polynomials of degree  $n$  [5], the Haar subspaces [7], the spaces of spline functions with fixed knots [3], the spaces of continuous functions obtained by pasting together Haar subspaces [1], and the spaces of generalized spline functions [16] are Chebyshev subspaces of  $C_1[0, 1]$ . The reason is that all these subspaces have a common feature— $A$ -property. The sufficiency of Theorem 1.1 was proved by Strauss [19] for  $K = [0, 1]$  and  $\omega \equiv 1$ . The general version of the sufficiency follows easily from his proof. The more difficult part of Theorem 1.1 is its necessity. When  $K = [0, 1]$  and  $G$  is a  $Z$ -subspace (i.e. no element of  $G$  vanishes on a nonempty open subset of  $K$ ), Havinson [4] proved that if  $G$  is a Chebyshev subspace of  $C_\omega(K)$  for every  $\omega \in W = W_B := \{\omega \in W_\infty : \inf \omega > 0\}$ , then  $G$  satisfies the Haar condition on  $(0, 1)$ . When  $W = W_B$ , the necessity of Theorem 1.1 was established by Kroó [8] for  $K = [0, 1]$ . Sommer [17] generalized Kroó's result for any  $K$ . Meanwhile, when  $W = W_C := \{\omega \in C(K) : \omega > 0 \text{ on } K\} \subset W_B$ , Pinkus [13] proved the necessity under the assumption that  $\mu(Z(g)) = \mu(\text{int } Z(g))$  for  $g \in G$ . Later, Kroó [9] showed that Pinkus's assumption can be removed. The present form of the necessity of Theorem 1.1 was proved by Schmidt [15] via the Liapounoff convexity theorem.

Even though  $A$ -property characterizes the Chebyshev subspaces with respect to varying weights, it is not easy to verify. However, when  $K \subset \mathbb{R}^1$ , the  $A$ -subspaces have easily verifiable characterizations. To state the results, we need some terminology and notation.

Recall that an  $n$ -dimensional subspace  $G$  of  $C[a, b]$  is called a weak Chebyshev subspace, if there do not exist  $g \in G$  and points  $a \leq t_0 < t_1 < \cdots < t_n \leq b$  such that  $g(t_{i-1}) \cdot g(t_i) < 0$  for  $1 \leq i \leq n$ . A finite-dimensional subspace  $G$  of  $C(K)$  is said to satisfy the Haar condition on  $B \subset K$  (or  $G|_B$  is a Haar subspace), if every nonzero element of  $G|_B$  has at most  $(\dim G|_B - 1)$  zeros. For any set  $A \subset [a, b]$  and  $F, H \subset C[a, b]$ , denote  $\text{bd } A :=$  the boundary set of  $A$ ;  $\text{int } A :=$  the interior of  $A$ ;  $Z(F) := \{t \in [a, b] : f(t) = 0 \text{ for all } f \in F\}$ ;  $\text{supp } F := [a, b] \setminus Z(F) = \{t \in [a, b] : f(t) \neq 0 \text{ for some } f \in F\}$ ;  $F|_A := \{f|_A : f \in F\}$ ;  $F(A) := \{f \in F : f|_A \equiv 0\}$ ;  $F \oplus H := \{f + h : f \in F \text{ and } h \in H\}$ .

**Theorem 1.2** (Pinkus [13]). *Suppose that  $G$  is a finite-dimensional subspace of  $C[a, b]$  and  $Z(G) \cap (a, b) = \emptyset$ . Then  $G$  is an  $A$ -subspace if and only if  $G$  satisfies the following conditions:*

- (1)  $G$  is a weak Chebyshev subspace;
- (2) There exist points  $a = c_0 < c_1 < \cdots < c_s = b$  with  $s \leq 2n - 2$  such that  $G|_{(c_{i-1}, c_i)}$  is a Haar subspace,  $1 \leq i \leq s$ ;

- (3)  $G([c_i, c_j]) = G([c_i, b]) \oplus G([a, c_j])$ ,  $0 \leq i < j \leq s$ ;  
 (4)  $G([a, c_i] \cup (c_j, b])$  is a weak Chebyshev subspace,  $0 \leq i < j \leq s$ .

**Theorem 1.3** (Pinkus [13 or 14]). Suppose that  $G$  is a finite-dimensional subspace of  $C[a, b]$ . Then  $G$  is an  $A$ -subspace if and only if, for any  $g \in G$ , the number of the connected components of the set  $\text{supp}(g)$  is no more than  $\dim G(\text{int } Z(g))$ .

**Theorem 1.4** (Kroó, Schmidt and Sommer [10]). Suppose that  $G$  is a finite-dimensional subspace of  $C[a, b]$  and  $Z(G) \cap (a, b) = \emptyset$ . Then  $G$  is an  $A$ -subspace if and only if  $G$  satisfies the following conditions:

- (1)  $G$  is a weak Chebyshev subspace;  
 (2) For  $a = t_0 < t_1 < \dots < t_n < t_{n+1} = b$  with  $\{t_i\}_1^n \subset Z(g)$  and  $g \in G \setminus \{0\}$ , there exists  $p \in G \setminus \{0\}$  such that  $(-1)^i \cdot p \geq 0$  on  $[t_i, t_{i+1}]$  for  $0 \leq i \leq n$ .

**Theorem 1.5** (Pinkus and Wajnryb [14]). Suppose that  $G$  is an  $A$ -subspace of  $C(K)$  with  $K \subset \mathbb{R}^1$ . Then there exist disjoint open intervals  $\{(a_i, b_i)\}_1^s$  such that

- (1)  $G = \bigoplus \sum_{i=1}^s G(K \setminus [a_i, b_i])$ ,  
 (2)  $G_i := G(K \setminus [a_i, b_i])|_{[a_i, b_i]}$  is an  $A$ -subspace for each  $1 \leq i \leq s$ , and  
 (3)  $Z(G_i) \cap (a_i, b_i) = \emptyset$  for each  $1 \leq i \leq s$ .

When  $K \subset \mathbb{R}^n$  ( $n > 1$ ), there are only some necessary conditions of  $A$ -subspaces (cf. [14]). Our main purpose of this paper is to simplify Pinkus's characterization given in Theorem 1.2. As a consequence we establish a relation between the existence of continuous metric selections for  $G$  and the  $A$ -property of  $G$ . Meanwhile, we show some interesting properties of weak Chebyshev subspaces.

Recall that the metric projection  $P_G$  from a normed linear space  $X$  to its subspace  $G$  is the following set-valued mapping:

$$P_G(f) := \{g \in G : \|f - g\| = d(f, G)\}, \quad \text{for } f \in X,$$

where  $\|\cdot\|$  denotes the norm on  $X$  and

$$d(f, G) := \inf\{\|f - p\| : p \in G\}.$$

A continuous mapping  $S$  from  $X$  to  $G$  is called a continuous selection for  $P_G$  if  $S(f) \in P_G(f)$  for every  $f \in X$ .

Then the main results in this paper can be summarized as follows:

**Theorem 1.6.** Suppose that  $G$  is a finite-dimensional subspace of  $C[a, b]$  and  $Z(G) \cap (a, b) = \emptyset$ . Then  $G$  is an  $A$ -subspace if and only if  $G$  is a weak Chebyshev subspace and  $G([c, d]) = G([a, d]) \oplus G([c, b])$  for any  $a < c < d < b$ .

**Theorem 1.7.** Suppose that  $G$  is a finite-dimensional subspace of  $C[a, b]$  and  $Z(G) \cap (a, b) = \emptyset$ . If  $P_G$  has a continuous selection in  $C[a, b]$ , then  $G$  is an  $A$ -subspace.



By a similar argument used by Sommer [16], we can prove that the weak Chebyshev property of  $G$  is inherited by some “special” subspaces of  $G$ .

**Theorem 2.1.** *If  $G$  is a weak Chebyshev subspace, then  $G([a, c) \cup (d, b])$  is also a weak Chebyshev subspace for any  $a \leq c < d \leq b$ .*

*Proof.* First fix  $a < e < b$ . Let  $\dim G|_{[a, e]} = r$ , so that  $\dim G([a, e]) = n - r$ . Then there exists  $g_1 \in G|_{[a, e]}$  with  $r$  equioscillations on  $(a, e)$  (i.e.,  $g_1(x_i) = (-1)^i$ ,  $1 \leq i \leq r$ , for some  $a < x_1 < x_2 < \dots < x_r < e$ ). If  $G([a, e])$  was not a weak Chebyshev subspace, then it would contain an element  $g_2^*$  with  $(n - r)$  sign changes in  $(e, b)$ . But then for  $\varepsilon$  small enough with proper sign,  $\varepsilon g_1 + g_2^*$  has  $n$  sign changes in  $(a, b)$ , contradicting the weak Chebyshev property of  $G$ . Analogously it can be shown that  $G([e, b])$  is a weak Chebyshev subspace. Theorem 2.1 follows from the identity  $G([a, c) \cup (d, b]) = (G([a, c))((d, b])$ .

*Remark.* Using similar arguments, we can prove the following statements about weak Chebyshev subspaces:

- (1) If  $G$  is a weak Chebyshev and  $\dim G|_{[c, d]}$  is even, then  $G([c, d])$  is a weak Chebyshev subspace.
- (2) Suppose that  $G$  is a weak Chebyshev and  $\dim G|_{[c, d]}$  is odd. Then  $G([c, d])$  is weak Chebyshev if  $G([c, d]) = G([a, d]) \oplus G([c, b])$ .
- (3) If  $G$  is an  $A$ -subspace, then  $G([c, d])$  is a weak Chebyshev subspace for any  $a \leq c < d \leq b$ .

The next theorem shows a decomposition property of certain subspaces of a weak Chebyshev subspace.

**Theorem 2.2.** *Let  $G$  be a weak Chebyshev subspace and  $G^* = G([a, c) \cup (d, b])$  with  $a \leq c < d \leq b$ . Then for any  $z \in Z(G^*) \setminus Z(G)$ ,  $G^* = G^*([a, z]) \oplus G^*([z, b])$ .*

*Proof.* Let  $\dim G^* = r$ . Without loss of generality we may assume  $c < z < d$ . Choose  $\{y_i\}_1^r \subset (c, z) \cup (z, d)$  and a basis  $\{g_i\}_1^r$  for  $G^*$  so that  $g_i(y_j) = \delta_{ij}$  ( $1 \leq i, j \leq r$ ). Since  $z \notin Z(G)$ , we may find  $\{y_i\}_{r+1}^n \subset (a, c) \cup \{z\} \cup (d, b)$ , where some  $y_j = z$ , and functions  $\{g_i\}_{r+1}^n \subset G$  so that  $g_i(y_j) = \delta_{ij}$  ( $r+1 \leq i, j \leq n$ ). For  $r+1 \leq i \leq n$  we replace  $g_i$  by  $g_i - \sum_{j=1}^r g_i(y_j) \cdot g_j$ . Thereby, we get  $g_i(y_j) = \delta_{ij}$  ( $1 \leq i, j \leq n$ ) and  $\{g_i\}_1^n$  is a basis for  $G$ . To verify the theorem we need to show that for  $1 \leq i \leq r$ , if  $y_i \in (c, z)$ , then  $g_i \equiv 0$  on  $[z, d]$  and if  $y_i \in (z, d)$ , then  $g_i \equiv 0$  on  $[c, z]$ . Let  $y_i \in (c, z)$  (the second case is similar). Then

$$g_i = D \begin{pmatrix} g_1, \dots, g_i, \dots, g_n \\ y_1, \dots, x, \dots, y_n \end{pmatrix}$$

and it follows by the weak Chebyshev property of  $G^*$  that  $g_i$  has weak sign changes on  $\{y_j\}_{j=1, j \neq i}^r$ . Also,  $\{y_j\}_{j=1, j \neq i}^n$  is a set of  $(n - 1)$  points in  $(a, b)$  and since  $G$  is a weak Chebyshev subspace, there exists  $g \in G \setminus \{0\}$  with weak sign change on  $\{y_j\}_{j=1, j \neq i}^n$ . Evidently,  $g = c \cdot g_i$  for some  $c \neq 0$ . As a result,

$g_i$  has weak sign changes on  $\{y_j\}_{j=1, j \neq i}^n$ . For  $z < x < d$ , the two weak sign change properties for  $g_i$  and the fact that  $y_j = z$  for some  $1 \leq j \leq n$ ,  $j \neq i$ , imply that  $g_i(x) = 0$ . So  $g_i \equiv 0$  on  $[z, d]$ .

Theorem 2.2 has a very interesting application. In the proof of Theorem 4.16 in [13], Pinkus applied the reasoning of case (1) to case (2) and case (3). But the reasoning of case (1) works in case (2) and case (3) only when  $Z(G([a, c_i) \cup (c_j, b])) \cap (c_i, c_j) = \emptyset$ , which is not true in general. However, Theorem 4.16 in [13] still holds, since our following corollary of Theorem 2.2 complements the gap in Pinkus's proof (cf. [13, p. 243]). The decomposition property shown in the following corollary is a property of  $A$ -subspaces (cf. [13, Proposition 4.9]).

**Corollary 2.1.** *Suppose that  $G$  is a weak Chebyshev subspace,  $Z(G) \cap (a, b) = \emptyset$  and  $G([c, d]) = G([a, d]) \oplus G([c, b])$  for any  $a < c < d < b$ . Then for any open set  $V$ , let  $\{V_i\}_1^r$  be the connected components of  $\text{supp } G(V)$ ,  $G(V) = \bigoplus_{i=1}^r G([a, b] \setminus V_i)$ .*

*Proof.* Let  $\{W_i\}_1^r$  be the connected components of the closure of  $\text{supp } G(V)$ . Then by the hypothesis,  $G(V) = \bigoplus_{i=1}^r G([a, b] \setminus W_i)$ . So it suffices to show that the conclusion in Corollary 2.1 holds for each  $G([a, b] \setminus W_i)$ .

Since  $W_i$  is closed and connected,  $W_i = [a_i, b_i]$  for some  $a \leq a_i < b_i \leq b$ . By Theorem 2.1,

$$G([a, b] \setminus W_i) = \bigoplus_{j=1}^s G([a, b] \setminus (x_{j-1}, x_j)),$$

where  $a_i = x_0 < x_1 < \cdots < x_s = b_i$  and  $\{x_j\}_1^{s-1} = Z(G([a, b] \setminus W_i)) \cap (a_i, b_i)$ . Since  $\{(x_{j-1}, x_j)\}_1^s$  are the connected components of  $\text{supp } G([a, b] \setminus W_i)$ , we complete the proof of Corollary 2.1.

*Remark.* Since Lemma 2.1 holds for general weak Chebyshev subspaces [2], so does every theorem and corollary in this section.

### 3. PROOF OF MAIN RESULTS

In this section we give proofs of Theorem 1.6, Theorem 1.7, and Theorem 1.8. By Theorem 2.1, statement (1) implies statement (4) in Theorem 1.2. Now we show that statements (1) and (3) imply statement (2) in Theorem 1.2. Thus, statements (1) and (3) in Theorem 1.2 are the characterization of  $A$ -subspaces  $G \subset C[a, b]$  with  $Z(G) \cap (a, b) = \emptyset$ . To prove Theorem 1.6 we need the following two lemmas.

**Lemma 3.1** (Stockenberg [18]). *If  $G$  is a weak Chebyshev subspace and  $g \in G$  with  $\text{int } Z(g) = \emptyset$ , then  $\text{card}\{(Z(g) \setminus Z(G)) \cap (a, b)\} \leq \dim G - 1$ .*

**Lemma 3.2** (Pinkus [13]). *If  $G([c, d]) = G([a, d]) \oplus G([c, b])$  for any  $a < c < d < b$ , then there exists a minimal set of knots  $a = x_0 < x_1 < \cdots < x_s = b$  with  $s \leq 2n - 2$  such that  $G|_{[x_{i-1}, x_i]}$  is a  $Z$ -subspace for each  $1 \leq i \leq s$ .*

*Proof of Theorem 1.6.*

*Necessity.* It follows from Theorem 1.2.

*Sufficiency.* By Theorem 1.2, Theorem 2.1 and Lemma 3.2, it suffices to show that  $G|_{(c,d)}$  is a Haar subspace if  $G|_{[c,d]}$  is a  $Z$ -subspace. By Lemma 2.2,  $G|_{[c,d]}$  is a weak Chebyshev subspace. Since  $Z(G) \cap (c, d) = \emptyset$  and  $G|_{(c,d)}$  is a  $Z$ -subspace, for any nonzero element  $g \in G|_{(c,d)}$ ,  $\text{int } Z(g) \cap (c, d) = \emptyset$ . By Lemma 3.1,  $\text{card}(Z(g)) = \text{card}(Z(g) \setminus Z(G|_{[c,d]})) \leq \dim G|_{(c,d)} - 1$ . So,  $G|_{(c,d)}$  is a Haar subspace. This completes the proof of Theorem 1.6.

Theorem 1.7 is a consequence of Theorem 1.6 and the following two lemmas.

**Lemma 3.3** (Li [11] or [12]). *Suppose  $\text{card}(\text{bd } Z(g)) \leq \dim G(\text{int } Z(g))$  for all  $g$  in  $G$ . Then  $G(V) \subset G([a, d]) \oplus G([c, d])$  for any open subset  $V \subset [a, b]$  and  $c, d \in Z(G(V))$  with  $c \neq d$ . In particular,  $G([c, d]) = G([a, d]) \oplus G([c, b])$  for any  $a < c < d < b$ .*

*Remark.* Sommer [16] also proved the above equality under slight different conditions. Pinkus [13] proved the same equality under the hypothesis that  $G$  is an  $A$ -subspace.

**Lemma 3.4** (Li [12]). *Suppose that  $G$  is a finite-dimensional subspace of  $C[a, b]$ . Then  $P_G$  has a continuous selection if and only if  $G$  is a weak Chebyshev subspace and  $\text{card}(\text{bd } Z(g)) \leq \dim G(\text{int } Z(g))$  for every  $g \in G$ .*

*Proof of Theorem 1.8.* Let  $g \in G \setminus \{0\}$ . If  $\text{int } Z(g) = \emptyset$ , then, by Lemma 3.1,  $g$  has at most  $(\dim G - 1) < \dim G = \dim G(\text{int } Z(g))$  zeros on  $(a, b)$ . Otherwise, by Theorem 2.2,

$$G(\text{int } Z(g)) = \bigoplus_{i=1}^r G([a, b] \setminus V_i),$$

where  $\{V_i\}_1^r$  are the connected components of  $\text{supp}(G(\text{int } Z(g)))$ . Obviously,

$$\dim G(\text{int } Z(g)) = \bigoplus_{i=1}^r \dim G([a, b] \setminus V_i) =: \sum_{i=1}^r r_i.$$

Since no element in  $G$  has two zero intervals,  $V_i \cap (a, b) = (a, z_i)$  or  $(z_i, b)$  for some  $a < z_i < b$ . Let  $B_i$  be the closure of  $V_i$ . Then  $\text{bd } B_i \cap (a, b)$  is a singleton. By Lemma 2.2 and Theorem 2.1,  $G([a, b] \setminus V_i)|_{B_i}$  is a weak Chebyshev subspace. By Lemma 3.1,  $\text{card}(Z(g) \cap V_i \cap (a, b)) \leq r_i - 1$ . Therefore,

$$\begin{aligned} \text{card}(\text{bd } Z(g) \cap (a, b)) &\leq \sum_{i=1}^r \text{card}(Z(g) \cap B_i \cap (a, b)) \\ &\leq \sum_{i=1}^r \{\text{card}(Z(g) \cap V_i \cap (a, b)) + \text{card}(\text{bd } B_i \cap (a, b))\} \\ &\leq \sum_{i=1}^r r_i = \dim G(\text{int } Z(g)). \end{aligned}$$

Moreover, the above inequality implies [12] that

$$\text{card}(\text{bd } Z(p)) \leq \dim G^*(\text{int } Z(p))$$

for any  $p \in G^*$ , where  $G^* = G|_{[c, d]}$  with  $a < c < d < b$ . By Lemma 2.2,  $G^*$  is a weak Chebyshev subspace. Thus, it follows from Lemma 3.4 that  $P_{G^*}$  has a continuous selection in  $C[c, d]$ .

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